How to Solve the Complete Cubic by the Method of 'Bases'

B. M. Shumyagskiy

Source: Tablitsy dlya Resheniya Kubicheskikh Uravneniy

Metodom Osnov /Tables for the Solution of Cubic

Equations by the Method of Bases_7.

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/Note: The following information came from a small hard-back book entitled 'Tablitsy dlya Resheniya Kubicheskikh Uravneniy Metodom Osnov' /Tables for the Solution of Cubic Equations by the Method of Bases7, by B. M. Shumyagskiy; State Press of Technical-Theoretical Literature, Moscow/Leningrad: 1950.

The information consists of the book's 'Contents', 'Foreword', and 'Explanation of Tables', with a sample of the tables.

Contents

Foreword

The present tables are intended for the solution of complete cubic equations by way of their reduction to the trinomial equations of special form. The idea of this

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method is contained in the author's article "Ischisleniye Shoyrosh" /"Calculus of 'Shoyrosh'"*7, published in Matematicheskiy Sbornik /Mathematical Symposium7, Volume 40' (1938), 3. /*Note: 'Shoyrosh' is an Ashkenazic Hebrew word for 'root'.7

It is assumed that during employment of these tables all computations will be conducted on the 'arithmometer'; but in cases not requiring great accuracy computations can be done on a slide-rule.

The author will be very grateful for any remarks, instruction, and advice, which should be directed to the address: Moscow, Orlikov 3, Gostekhizdat.

Explanation of the Tables

Section 1.

In these tables the roots of the equation

$$z^3 + Az - A = 0 \tag{1}$$

are given for various values of A. For greater detail concerning the properties of trinomial equations of the type $x^n + Ax - A = 0$, see the author's article entitled "Ischisleniye Shoyrosh" /"Calculus of 'Shoyrosh'"*7, published in Matematicheskiy Sbornik /Mathematical Symposium7, Volume 40' (1938), 3. /*Note: 'Shoyrosh is Ashkenazic for 'root'.7 To each value of A in the tables correspond three roots, which we eill call the third-order bases ('roots') of A and designate by the symbol

$$z = \sqrt[3]{A} \tag{2}$$

Section 2. The Contents of the Tables

Table I contains the bases (roots) of equation (1), corresponding to positive values of A from $8 \cdot 10^{-9}$ to 300. To each value of A corresponds one real basis (Root) and two-complex-conjugate bases (-roots).

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In the table are given the values of the real basis (root), designated by z_1, and the values of the quotient from dividing the absolute value of the coefficient of i (the imaginary part) of the complex roots by the absolute value of the real root z_1; this quotient is designated in the table by alpha α.

If the value of z_1 is known, then the value $z_{2,3}$ is expressed thus:

$$z_{2,3} = -\frac{1}{2} z_1 \pm i z_1 \alpha \tag{3}$$

The preceding equation follows from the fact that the sum of all three roots of equation (1) equal 0.

In Table II are given the bases ('roots') of negative values of A greater than -6.75 to the number -0.000,000,008 inclusively.

To these values of A also correspond one real negative basis (root) and two complex conjugate bases (roots). Just as in Table I, Table II also gives the real negative bases (root) z_1 and value of α (as above: $z_{2,3} = -\frac{1}{2} z_1 \pm i z_1 \alpha$).

Table III contains negative values of A from -6.75 to -300. To these values of A correspond three real bases (roots): one negative and two positive.
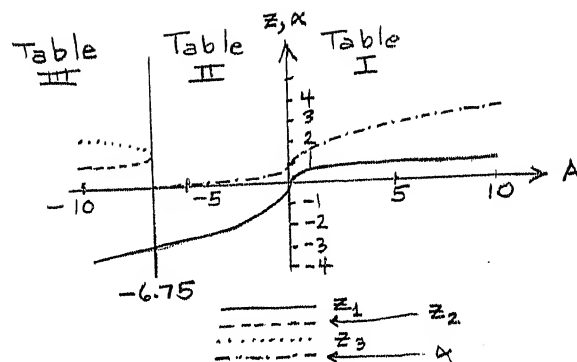
In the table the negative basis (root) is designated by z_1, and the smaller positive basis (root) is designated by z_2.

Correspondingly, the table consists of two parts: Table IIIa contains values of z_1 (negative), and Table IIIb contains values of z_2 (positive). The third root is found from the formula $z_3 = -(z_1 + z_2)$.

The distribution of roots with respect to the tables is illustrated by the diagram:

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The tables give an accuracy to four decimal places for the real bases (roots) and to three decimal places for complex bases (roots).

The 4th and 5th places for the first case and the 4th place for the second case are found by linear interpolation. In most of the cases the interpolation permits one to find 5 accurate places of the real basis (root). The pages where linear interpolation gives only four places (but not five) for the real basis (root) are noted with an asterisk placed in the left upper corner of the page.

Section 3. Solution of the Complete Cubic Equation

First we shall show how to solve the trinomial cubic equation and afterwards the complete cubic.

THE equation (4)

$$y^3 + py + q = 0$$

can be reduced to an equation of the form (1)

$$z^3 + Az - A = 0$$

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to do this it is necessary to set in equation (4)

$$y = - qz/p \qquad (5)$$

We then obtain

$$z^3 + \frac{p^3}{q^2} z - \frac{p^3}{q^2} = 0 .$$

The equation obtained is an equation of form (1), in which $A = p^3/q^2$.

In our symbols the equation is solved:

$$y = - \frac{q}{p} \sqrt[3]{\frac{p^3}{q^2}} . \qquad (6)$$

Let the following equation be given

$$x^3 + ax^2 + bx + c = 0 \qquad (7)$$

The equation given will be solved in this order:

1. Eliminating the second term (ax^2) in this equation (7), we reduce it to the form (4).

2. The equation of form (4) is reduced to an equation of the form (1) and solved in accordance with formula (6).

Setting in equation (7)

$$x = y - a/3,$$

we obtain

$$y^3 + (b - \frac{a^2}{3})y + (\frac{2a^3}{27} - \frac{ab}{3} + C) = 0 ; \qquad (8)$$

this is an equation of form (4), in which we have

$$p = b - \frac{a^2}{3} ; \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c .$$

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Setting in equation (8) y = - qz/p, we obtain

$$z^3 + \frac{[3(3b-a^2)]^3}{(2a^3-9ab+27c)^2} z - \frac{[3(3b-a^2)]^3}{(2a^3-9ab+27c)^2} = 0 \; ; \tag{9}$$

thus we have arrived at equation (1), in which

$$A = \frac{[3(3b-a^2)]^3}{(2a^3-9ab+27c)^2} \; . \tag{10}$$

Now let the value of the basis (root) in equation (9) be known to us. Let us find the value of the root of the original equation (7).

We have

$$y = - qz/p,$$

$$x = y - a/3 = - qz/p - a/3 \tag{11}$$

Using our symbol, we can write, by setting

$$A = \frac{[3(3b-a^2)]^3}{(2a^2-9ab+27c)^2} \quad\text{and}\quad z = \sqrt[3]{A} \; , \quad\text{as follows:}$$

$$x = -\frac{1}{3}\left[\frac{2a^3-9ab+27c}{3(3b-a^2)} \cdot \sqrt[3]{\frac{[3(3b-a^2)]^3}{(2a^3-9ab+27c)^3}} + a \right]. \tag{12}$$

For b = 0, we obtain

$$x = -\frac{1}{3}\left[\frac{2a^3+27c}{3a^2} \cdot \sqrt[3]{\frac{27a^6}{(2a^3+27c)^2}} + a \right]. \tag{13}$$

For a = 0, we obtain

$$x = -\frac{c}{b} \sqrt[3]{\frac{b^3}{c^2}} \; . \tag{14}$$

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For an equation of the form $mx^3 + mx + k = 0$,

we get

$$x = -\frac{k}{n} \cdot \sqrt[3]{\frac{n^3}{k^2 m}} \,.$$

(15)

Thus the solution of a cubic equation of form (7) reduces to the solution of an equation of the form

$$z^3 + Az - A = 0,$$

whose roots we shall call the third-order <u>bases</u> ('roots') of A.

Let us consider the two cases separately:

1. $A \leqslant -6.75$

2. $A > -6.75$

If $A \leqslant -6.75$, then all three bases ('roots') of A (namely, z_1, z_2, z_3) are real.

In this case we first find A according to formula (10); then we find z_1, z_2, z_3 from the tables. Further, we find according to one of the formulas (12)-(15) (depending upon the form of the equation) the three roots of the original equation (7).

If $A > -6.75$, then one of the bases ('roots') z_1 is real, but the other two bases are complex-conjugate. In this case we first find the real root x_1 of equation (7) in the same order as in the first case.

In order to find the complex roots of equation (7) we use formula (3):

$$z_{2,3} = -\frac{z_1}{2} \pm i z_1 \alpha \,.$$

Hence, according to formula (5) we have:

$$y_{2,3} = \frac{q}{p} \cdot \frac{z_1}{2} + i \frac{q}{p} \cdot z_1 \alpha = -\frac{y_1}{2} \pm i y_1 \alpha \,.$$

(16)

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Finally, $$x_{2,3} = -\frac{x_1 + a}{2} \pm i\left(x_1 + \frac{a}{3}\right)\alpha.$$ (17)

If $A > -6.75$, then we find according to Table I (if $A > 0$) or Table II (if $-6.75 < A < 0$) the values of z_1 and alpha α.

From z_1 we find the first root (x_1) of equation (7). Further, we find the values of $y_{2,3}$ or $x_{2,3}$ according to formula (16) or (17). Thus:

for equation $y^3 + py + q = 0$,

we have $\quad y_{2,3} = -y_1/2 \pm iy_1\alpha$;

for equation $x^3 + ax^2 + bx + c = 0$,

we have $\quad x_{2,3} = -\frac{x_1-a}{2} \pm i\left(x_1 + \frac{a}{3}\right)\alpha.$

For $/A/ > 300$ or $/A/ < 8\cdot10^{-9}$ we can use the following approximate formulas:

1. If $A > 300$, then we have: $\quad z_1 = \dfrac{A+2}{A+3}$;

  (18)

$$z_{2,3} = -\frac{z_1}{2} \pm \sqrt{-A - 3\left(\frac{z_1}{2}\right)^2}.$$ (19)

2. If $A < -300$, then we have:

$$z_2 = \frac{A+2}{A+3} ;$$ (20)

$$z_{3,1} = -\frac{z_2}{2} \pm \sqrt{-A - 3\left(\frac{z_2}{2}\right)^2}.$$ (21)

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3. If $|A| < 10^{-9}$, then we have :

$$z_{1,2,3} = \sqrt[3]{A}$$

(22)

We shall show how to use the tables by examples

Example 1. $x^3 + 0.9156x - 1.91 = 0$

We solve according to formula (14):

$$x_1 = \frac{1.91}{0.9156} \sqrt[3]{\frac{0.9156^3}{1.91^2}} = \frac{1.91}{0.9156} \cdot \sqrt[3]{0.21041} \ .$$

In the example given, **A** is a positve number; it has one real positive basis ('root') and two complex bases ('roots'). The bases ('roots'(of positive numbers (from 0 to 300) are found in Table I.

On page 20 we find the real basis ('root') z_1 and coefficient α in the imaginary part of the complex roots. We find three places immediately from the table. The remaining places are found by linear interpolation:

$$\begin{array}{lll}
0.21041 & 0.47800 & 1.2906 \\
0.20923 & \underline{\quad 69} & 15 \cdot 0.69 \dots \underline{\quad 10} \\
\underline{\quad 118\quad} \approx 0.69, \; z_1 = 0.47869 & & = 1.2916 \\
\quad 172 & &
\end{array}$$

Now we find the value of the roots, the complex being found according to formula (16);

$$x_1 = \frac{1.91}{0.9195} \cdot (0.47869) = 0.99858 \leftarrow$$

$$\hookleftarrow \text{(with an accuracy to fifth place);}$$

$$x_{2,3} = \frac{1}{2} \, 0.99858 \pm i \cdot 0.99858 \, \alpha \ .$$

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$$= -0.49929 \pm i \cdot 0.99858 \cdot 1.2916$$

$$= -0.49929 \pm i \cdot 1.2897$$

/Note: Three other examples are given in the original; namely:

$$716 x^3 - 512x + 467 = 0;$$

$$x^3 - 2x^2 + 3x - 5 = 0;$$

$$x^3 - 5x + 3 = 0.7$$

/Example of a page from Table I.7

Table I. For A from 0.11682 to 0.23068   (p. 20)

A	d	z_1	α		A	d	z_1	α
0.11682		0.410	1.2020		0.16568		0.450	1.2522
11787	105					141		
11787		411	2032					
	107							
11894		412	2044					
	107							
12001		413	2056					
	108							
12109		414	2069					
0.12218		0.415	1.2081					
	1 109	0.415						

. . . .

/Note: Table II is similar to
table I in format.7

/etc; there are 6 x 8 = 48 lines
like this per half side of page.7

/Example of a page from Table IIIa.7

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Table IIIa. For A from - 10.39408 to - 10.90654 (p 80)

A	d	z_1		A	d	z_1
- 10.39408		- 3.640		- 10.64872		3.680
	633				641	

. . . .

$\underline{/}$etç; there are 6 x 8 = 48 $\underline{/}$etc; there are 6 x 8 = 48

similar lines.$\underline{/}$ similar lines.$\underline{/}$

$\underline{/}$Note: Table IIIb is like Table IIIa in format.$\underline{/}$

- E N D -